ON SUMS OF FOURIER COEFFICIENTS OF CUSP FORMS

Aleksandar Ivić

The aim of this note is to evaluate asymptotically the sum

$$F(x) \;:=\; \sum_{n\le x} f(n^2) \tag{1}$$

in case $f(n)$ is the Fourier coefficient of a holomorphic or non-holomorphic cusp form. We shall first deal with the latter case, which is more complicated. Let as usual $\{\lambda_j = \kappa_j^2 + \frac14\} \cup \{0\}$ be the discrete spectrum of the non-Euclidean Laplacian acting on $SL(2,\mathbb{Z})-$automorphic forms. Further let $\rho_j(n)$ denote the n-th Fourier coefficient of the Maass wave form $\varphi_j(z)$ corresponding to the eigenvalue λ_j to which the Hecke series

$$H_j(s) \;=\; \sum_{n=1}^\infty t_j(n)n^{-s} \qquad (\Re\mathrm{e}\, s > 1)$$

is attached (see Y. Motohashi [6] for an extensive account). For every $n\in\mathbb{N}$ we have $\rho_j(n) = \rho_j(1)t_j(n)$, so that one may consider sums of $t_j(n)$ instead of sums of $\rho_j(n)$. In [4] T. Meurman and the author proved that, for $\kappa_j \le x^{1-\alpha}$, we have uniformly in κ_j

$$\sum_{n\le x} t_j^2(n) \;=\; \frac{12x}{\pi^2\alpha_j} + O_\varepsilon(x^\varepsilon R(x)) \tag{2}$$

with

$$R(x) \;:=\; \kappa_j^{\frac{1}{2-2\alpha}}x^{\frac12} + \min\left(\kappa_j^{\frac{1+10\alpha}{3+6\alpha}}x^{\frac12} + x^{\frac{3+6\alpha}{5}},\, \kappa_j^{-1}x^{1+2\alpha}\right). \tag{3}$$

Here as usual

$$\alpha_j \;=\; \frac{|\rho_j(1)|^2}{\cosh(\pi\kappa_j)}, \tag{4}$$

and α (≥ 0) is the constant for which

$$t_j(n) \;\ll_\varepsilon\; n^{\alpha+\varepsilon} \tag{5}$$

holds uniformly in κ_j. Moreover, ε denotes arbitrarily small positive constants, not necessarily the same ones at each occurrence, while $\ll_\varepsilon$ means that the $\ll$-constant depends on ε. The accent in (2)-(3) is on uniformity in κ_j, since in many applications κ_j may vary with x. It is known that (5) holds with $\alpha \le \frac{5}{28}$ (see D. Bump et al. [1]). In what concerns the order of α_j, we have

$$\kappa_j^{-\varepsilon} \;\ll_\varepsilon\; \alpha_j \;\ll_\varepsilon\; \kappa_j^{\varepsilon}. \tag{6}$$

2001 *Mathematics Subject Classification.* Primary 11F72, Secondary 11M06.

Key words and phrases: Fourier coefficients of cusp forms, Ramanujan-Petersson conjecture, Riemann zeta-function

The lower bound in (6) was proved by H. Iwaniec [5], and the upper bound by Hoffstein–Lockhart [2].

It seems that there are no upper bounds in the literature for the sum $F(x)$ in (1) when $f(n) = t_j(n)$. We shall prove

THEOREM 1. *For* $\kappa_j \le x^c$,

$$0 \;<\; c \;<\; \min\left(\frac{3+6\alpha}{2+20\alpha},\, 1-\alpha\right) \tag{7}$$

and a suitable constant $A > 0$ *we have, uniformly in* κ_j,

$$\sum_{n\le x} t_j(n^2) \;\ll\; \alpha_j^{-1} x \exp\left(-A\log^{3/5}x(\log\log x)^{-1/5}\right). \tag{8}$$

Corollary 1. With the value $\alpha \le \frac{5}{28}$ it follows that (8) holds uniformly in κ_j for $\kappa_j \le x^c$, and any constant c satisfying $0 < c < \frac{57}{78}$.

Corollary 2. If the Ramanujan-Petersson conjecture that $\alpha = 0$ is true, then (8) holds uniformly in κ_j for $\kappa_j \le x^c$, and any constant $0 < c < 1$.

Remark 1. The negative exponential in (8) comes from the sharpest known error term in the prime number theorem in the form (see e.g., [3, Chapter 12])

$$\sum_{n\le x} \mu(n) \;\ll\; x\exp\left(-C\log^{3/5}x(\log\log x)^{-1/5}\right) \qquad (C > 0). \tag{9}$$

Sharper forms of the prime number theorem, which would follow from a better zero-free region for the Riemann zeta-function $\zeta(s)$ than the one that is currently known (see [3, Chapter 6]), would therefore lead to a better estimate than (8).

Remark 2. By a result of T. Meurman and the author [4] one has

$$\sum_{n\le x} t_j(n) \;\ll_\varepsilon\; \kappa_j^{1+\varepsilon}$$

uniformly for $\sqrt{x} < \kappa_j \le x$, which may be compared to the bound in (8).

Remark 3. The oscillatory nature of the function $t_j(n)$ accounts for the lack of a main term in (8). However, if one looks at the problem of evaluating $F(x)$ in (1) when $f(n) = d(n)$, the number of divisors of n, then there will be a main term in the corresponding formula for the summatory function. Namely the function $d(n^2)$ is generated by $\zeta^3(s)/\zeta(2s)$, which has a pole of order three at $s = 1$. Consequently we have (see [3, eq. (14.29)]), for suitable constants $B_1\,(>0), B_2, B_3, C\,(>0)$,

$$\sum_{n\le x} d(n^2) = x(B_1\log^2 x + B_2\log x + B_3) + O\left(x\exp\left(-C\log^{3/5}x(\log\log x)^{-1/5}\right)\right).$$

Proof of Theorem 1. From the multiplicative property (see [6, eq. (3.2.8)])

$$t_j(mn) \;=\; \sum_{d|(m,n)} \mu(d) t_j\left(\frac{m}{d}\right) t_j\left(\frac{n}{d}\right),$$

where $\mu(n)$ is the Möbius function, one has

$$t_j(n^2) \;=\; \sum_{d|n}\mu(d)t_j^2\left(\frac{n}{d}\right). \tag{10}$$

Therefore (10) gives

$$\begin{aligned}\sum_{n\le x} t_j(n^2) &= \sum_{mn\le x}\mu(m)t_j^2(n)\\ &= \sum_{m\le\sqrt{x}}\mu(m)\sum_{n\le x/m}t_j^2(n) + \sum_{n\le\sqrt{x}}t_j(n^2)\sum_{\sqrt{x}<m\le x/n}\mu(m)\\ &= \sum\nolimits_1 + \sum\nolimits_2,\end{aligned}$$

say. We set for brevity

$$\eta(x) \;=\; (\log x)^{3/5}(\log\log x)^{-1/5},$$

and let $0<\beta<1$, $\beta=\beta(\alpha)$ be such a number for which (2)-(3) give

$$\sum_{n\le x} t_j^2(n) = \frac{12x}{\pi^2\alpha_j} + O(x^\beta) \tag{11}$$

uniformly for $\kappa_j \le x^c$. If C denotes generic positive constants, then

$$\begin{aligned}\sum\nolimits_1 &= \frac{12x}{\pi^2\alpha_j}\sum_{m\le\sqrt{x}}\frac{\mu(m)}{m} + O\Big(\sum_{m\le\sqrt{x}}\left(\frac{x}{m}\right)^\beta\Big)\\ &= \frac{12x}{\pi^2\alpha_j}\sum_{m>\sqrt{x}}\frac{\mu(m)}{m} + O(x^\beta x^{\frac{1-\beta}{2}})\\ &\ll \alpha_j^{-1}x\mathrm{e}^{-C\eta(x)},\end{aligned}$$

since in view of $0<\beta<1$ and the upper bound in (6) we have

$$x^{\frac{1+\beta}{2}} \;\ll\; \alpha_j^{-1}x\mathrm{e}^{-C\eta(x)}.$$

Here we also used (9), partial summation and the well-known fact that

$$\sum_{m=1}^{\infty}\frac{\mu(m)}{m} \;=\; 0$$

to deduce that

$$\sum_{m>\sqrt{x}}\frac{\mu(m)}{m} \;\ll\; \exp(-C\eta(x)). \tag{12}$$

We also have, on using (9) and (11),

$$\begin{aligned}\sum\nolimits_2 &\ll \sum_{n\le\sqrt{x}}t_j^2(n)\left(\frac{x}{n}\mathrm{e}^{-C\eta(x/n)} + \sqrt{x}\mathrm{e}^{-C\eta(x)}\right)\\ &\ll \mathrm{e}^{-C\eta(x)}\Big(x\sum_{n\le\sqrt{x}}\frac{t_j^2(n)}{n} + \sqrt{x}\sum_{n\le\sqrt{x}}t_j^2(n)\Big)\\ &\ll \alpha_j^{-1}x\mathrm{e}^{-C\eta(x)}.\end{aligned}$$

Now note that for $\kappa_j \le x^c$, $c < 1-\alpha$ we have

$$\kappa_j^{\frac{1}{2-2\alpha}} x^{\frac12} \;\le\; x^\beta, \quad \beta = \frac{c}{2-2\alpha} + \frac12 \quad (<1).$$

Moreover

$$\frac{1+10\alpha}{3+6\alpha} c \;<\; \frac12 \quad \text{for} \quad c < \frac{3+6\alpha}{2+20\alpha},$$

and $\frac{3+6\alpha}{5} < 1$. Hence (11) is satisfied if $\kappa_j \le x^c$ and c is any constant satisfying (7). This finishes the proof of Theorem 1.

The foregoing analysis may be applied to the case of a holomorphic cusp form

$$\varphi(z) \;=\; \sum_{n=1}^\infty a(n) \mathrm{e}^{2\pi i n z}$$

of weight κ with respect to $SL(2,\mathbb{Z})$. If $\varphi(z)$ is a normalized eigenform, i.e., an eigenfunction with respect to all Hecke operators and satisfies $a(1) = 1$, then all $a(n) \in \mathbb{R}$. We have (see e.g., R.A. Rankin [7] and [8])

$$\sum_{n=1}^\infty a(n) n^{-s} = \prod_p (1 - a(p)p^{-s} + p^{\kappa-1-2s})^{-1} \quad (\sigma > \tfrac12(\kappa+1)),$$

and

$$a(m)a(n) \;=\; \sum_{d|(m,n)} d^{\kappa-1} a\left(\frac{mn}{d^2}\right) \qquad (m,n \in \mathbb{N}). \tag{15}$$

If we introduce the "normalized" function $\tilde a(n)$, namely

$$a(n) \;=\; \tilde a(n) n^{\frac{\kappa-1}{2}}, \tag{16}$$

then we can write (15) as

$$\tilde a(m)\tilde a(n) \;=\; \sum_{d|(m,n)} \tilde a\left(\frac{mn}{d^2}\right) \qquad (m,n \in \mathbb{N}). \tag{17}$$

When $m = n$, (17) gives by the Möbius inversion formula

$$\tilde a(n^2) \;=\; \sum_{d|n} \mu(d) \left(\tilde a(\frac{n}{d})\right)^2, \tag{18}$$

which is analogous to (10). If we use (18), the Rankin-Selberg formula

$$\sum_{n\le x} \tilde a^2(n) \;=\; Cx + O(x^{3/5}) \qquad (C>0),$$

then by employing the method of proof of Theorem 1 we shall obtain

THEOREM 2. *We have*

$$\sum_{n\le x} \tilde a(n^2) \;\ll\; x \exp\Bigl(-C(\log x)^{3/5} (\log\log x)^{-1/5}\Bigr) \qquad (C>0).$$

From Theorem 2 we obtain by partial summation, on using (16),

Corollary 3. *We have*

$$\sum_{n\le x} a(n^2) \;\ll\; x^\kappa \exp\Bigl(-C(\log x)^{3/5} (\log\log x)^{-1/5}\Bigr) \qquad (C>0).$$

References

[1] D. Bump, W. Duke, J. Hoffstein and H. Iwaniec, An estimate for Fourier coefficients of Maass wave forms, Duke Math. J. **66**(1992), 75-81.
[2] J. Hoffstein and P. Lockhart, Coefficients of Maass forms and the Siegel zero, Annals of Math. **140**(1994), 161-176.
[3] A. Ivić, The Riemann zeta-function, John Wiley & Sons, New York, 1985.
[4] A. Ivić and T. Meurman, Sums of coefficients of Hecke series, Acta Arith. **68**(1994), 341-368.
[5] H. Iwaniec, Small eigenvalues of the Laplacian for $\Gamma_0(N)$, Acta Arith. **56** (1990), 65-82.
[6] Y. Motohashi, Spectral theory of the Riemann zeta-function, Cambridge University Press, Cambridge, 1997.
[7] R.A. Rankin, Modular forms and functions, Cambridge University Press, Cambridge, 1977.
[8] R.A. Rankin, Fourier coefficients of cusp forms, Math. Proc. Camb. Phil. Soc. **100**(1986), 5-29.

Aleksandar Ivić
Katedra Matematike RGF-a
Universiteta u Beogradu, Đušina 7,
11000 Beograd, Serbia (Yugoslavia)
e-mail: `aivic@matf.bg.ac.yu, aivic@rgf.bg.ac.yu`